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A theoretical study of viscous effects in peristaltic pumping

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Intuition and previous results suggest that a peristaltic wave tends to drive the mean flow in the direction of wave propagation. New theoretical results indicate that, when the viscosity of the transported fluid is shear-dependent, the direction of mean flow can oppose the direction of wave propagation even in the presence of a zero or favourable mean pressure gradient. The theory is based on an analysis of lubrication-type flow through an infinitely long, axisymmetric tube subjected to a periodic train of transverse waves. Sample calculations for a shear-thinning fluid illustrate that, for a given waveform, the sense of the mean flow can depend on the rheology of the fluid, and that the mean flow rate need not increase monotonically with wave speed and occlusion. We also show that, in the absence of a mean pressure gradient, positive mean flow is assured only for Newtonian fluids; any deviation from Newtonian behaviour allows one to find at least one non-trivial waveform for which the mean flow rate is zero or negative. Introduction of a class of waves dominated by long, straight sections facilitates the proof of this result and provides a simple tool for understanding viscous effects in peristaltic pumping.

1. Introduction

1.1. *Purpose*

Peristaltic pumping is the transport of fluid by a wave of contraction and/or expansion propagating along the walls of a tube or channel. The motion of the walls traditionally has been understood to be primarily transverse to the long axis of the conduit. The mean flow rate produced by such a device depends on the shape and speed of the wave, the properties of the fluid, and the pressure difference across which the pump is operating. Despite these complexities, it seems intuitively clear that a peristaltic wave should tend to drive the mean flow in the direction of wave propagation. To our knowledge, no previous theoretical or experimental findings dispute this notion.

In this paper we present theoretical results indicating that a transverse peristaltic wave moving in one direction can drive the mean flow in the opposite direction. (This phenomenon is not to be confused with reflux, which refers to the negative mean axial velocities of individual fluid elements.) The theory is based on an analysis of lubrication-type flow through an infinitely long, axisymmetric tube subjected to a periodic train of transverse waves. We make two assumptions regarding the constitutive behaviour of the transported fluid: the shear stress depends only on the shear rate, and non-Newtonian normal stresses are negligible.

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Sample calculations in §2 for a shear-thinning fluid demonstrate the following: the direction of mean flow can oppose the direction of wave propagation even in the presence of a mean pressure gradient that favours positive flow; for a given waveform, the sense of the mean flow can depend on the rheology of the fluid; and the mean flow rate need not increase monotonically with wave speed and occlusion. In §3, a more general lubrication-based analysis reveals that, in the absence of a mean pressure gradient, positive mean flow is assured for all waveforms only when the fluid is Newtonian over the entire range of permissible shear rates. In other words, if the transported fluid exhibits any deviations from Newtonian behaviour, one can find at least one non-trivial waveform for which the mean flow rate is zero or negative. Introduction of a class of waves dominated by long, straight sections facilitates the proof of this result and provides a simple tool for understanding viscous effects in peristaltic pumping.

1.2. *Related work*

Early theoretical work on peristaltic transport, notably that of Fung & Yih (1968), was concerned primarily with inertia-free, Newtonian flows driven by sinusoidal transverse waves of small amplitude. Shapiro, Jaffrin & Weinberg (1969) were among the first to present closed-form solutions for waves of long wavelength and arbitrary amplitude. They also derived conditions for the presence of closed streamlines, called trapping, and negative mean Lagrangian axial velocities of individual fluid elements, known as reflux. Jaffrin & Shapiro (1971) have summarized the early literature. Much subsequent work has been devoted to relaxing the original assumptions regarding flow geometry and the properties of the transported fluid. Numerical investigations by Brown & Hung (1977), Pozrikidis (1987), and Takabatake, Ayukawa & Mori (1988), among others, have revealed the effects of fluid inertia and wall curvature and alignment on peristaltic flow patterns and pumping characteristics.

Brasseur, Corrsin & Lu (1987) considered the influence of a distinct, Newtonian peripheral fluid layer in connection with pumping in physiological systems. The complex rheology of biological and physiological flows has also motivated a number of studies involving non-Newtonian fluids. The power-law model was used by Raju & Devanathan (1972), Picologlou, Patel & Lykoudis (1973), and Shukla & Gupta (1982) to investigate shear-thinning and shear-thickening effects. Becker (1980) presented an analysis, based on a geometry equivalent to one wavelength of the SSD wave introduced in §3.1, for fluids with shear-dependent viscosity and computed pumping characteristics for a Prandtl–Eyring fluid. Raju & Devanathan (1974) and Böhme & Friedrich (1983) probed the effects of viscoelasticity. Siddiqui, Provost & Schwarz (1991) used the second-order fluid model to study the effects of normal stresses in slow non-Newtonian flows.

The present work follows in the tradition of Shapiro, Jaffrin & Weinberg (1969) and others who have sought to understand viscous effects in peristalsis by analysing Newtonian and non-Newtonian flows in the lubrication limit. In contrast with these earlier works, however, our main conclusions are not based on a particular constitutive model. Rather, they apply to all fluids with shear-dependent viscosity, subject to the assumptions discussed in §1.4.

1.3. *Basic concepts and terminology*

Peristaltic flow is inherently unsteady in a frame of reference that is fixed in space. However, when a spatially periodic peristaltic wave propagates axially at a constant speed along an infinitely long tube, the flow is steady in a frame of reference that translates along with the wave (Shapiro *et al.* 1969). The fixed and moving frames of reference are called the *laboratory frame* and the *wave frame*, respectively. For quantities that depend upon the frame of reference, capital and lower-case letters denote values relative to the laboratory frame and the wave frame, respectively. The axial coordinates in the two frames transform as $z = Z - ct$, where c is the wave speed, and t is time. Axial velocities are related by $u = U - c$. In each frame of reference, velocities (and hence flow rates) are defined to be positive in the direction in which the axial coordinate (z or Z) increases. The wave speed is positive by convention.

Integration of u over a cross-section of the tube yields the relationship between the volumetric flow rates in the wave and laboratory frames,

$$q = Q - Ac, \tag{1}$$

where $A = A\{z\}$ is the cross-sectional area of the tube. Solving (1) for Q and averaging over one period gives the time-mean flow rate in the laboratory frame,

$$\bar{Q} = q + \bar{A}c, \tag{2}$$

where the overbar denotes an average over one period of the wave. (We note here for future reference that, for any quantity G,

$$\bar{G} = \frac{1}{\lambda}\int_0^\lambda G\{z\}\,\mathrm{d}z, \tag{3}$$

that is, the time mean over one period in the laboratory frame is equal to the spatial average over one wavelength, λ, in the wave frame.) Elimination of q between (1) and (2) yields

$$Q = \bar{Q} + c(A - \bar{A}). \tag{4}$$

As a rule, the flow rate generated by a peristaltic device is a function of the mean pressure gradient against which it operates. For a periodic wave of wavelength λ, the mean pressure gradient is $\Delta P_\lambda/\lambda$, where ΔP_λ denotes the pressure change over one wavelength. The terms *adverse* and *favourable* are used to describe positive and negative pressure gradients, respectively. *Free pumping* refers to the case $\Delta P_\lambda = 0$ (or, equivalently, $\Delta P_\lambda/\lambda = 0$), while *positive free pumping* implies additionally that $\bar{Q} > 0$. Note that positive flow rates are, by definition, in the direction of wave propagation. Throughout the remainder of the paper, all references to flows and flow rates will refer to the laboratory frame, unless specified otherwise.

1.4. *Main assumptions*

The results presented in this work pertain to peristaltic transport of fluid through an infinitely long, axisymmetric tube by a periodic transverse wave of vanishingly small slope. We assume that the flow is free of inertial effects and that non-Newtonian normal stresses are negligible. Under these conditions, the governing equations, expressed here in cylindrical coordinates (r, z), are similar to those found in lubrication theory. In general, the constitutive behaviour of the transported fluid is specified by relating the shear stress τ_{rz} to the deformation history. Because the flow through any given cross-section is essentially viscometric, we assume that the shear stress is a function of the shear rate only,

$$\tau_{rz} = f\{\kappa\}, \tag{5}$$

where $\kappa \equiv \partial u/\partial r$. Implicit in (5) is the assumption that fluid memory effects are negligible. We further assume that f is a continuously differentiable, strictly increasing (and therefore invertible) function with a bounded first derivative, and (following Coleman, Markovitz & Noll 1966) that it is an odd function that takes the same sign as its argument.

1.5. *Lubrication analysis*

Under the assumptions of the previous section, the momentum equation in cylindrical coordinates (r, z) reduces to

$$\frac{\mathrm{d}P}{\mathrm{d}z} = \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}). \tag{6}$$

Coleman, Markovitz & Noll (1966) solved (6), with the constitutive relation (5), within the context of steady, viscometric, Poiseuille flow. Their analysis is applicable to the present problem and is reproduced below with appropriate modifications.

Substituting (5) into (6) and integrating, we get

$$\frac{\partial u}{\partial r} = f^{-1}\left\{\tfrac{1}{2}r\frac{\mathrm{d}P}{\mathrm{d}z}\right\}. \tag{7}$$

The wave-frame flow rate is given by

$$q = 2\pi\int_{r=0}^{r=h} ru\{r\}\,\mathrm{d}r, \tag{8}$$

where $h\{z\} \equiv (A\{z\}/\pi)^{1/2}$ is the tube radius. Integrating (8) by parts, with the wave-frame boundary condition $u = -c$ at $r = h\{z\}$, we find

$$q = -cA - \pi\int_{r=0}^{r=h} r^2\frac{\partial u}{\partial r}\,\mathrm{d}r. \tag{9}$$

Substituting (7) into (9) and invoking (1), we obtain the relationship between the laboratory-frame flow rate and the pressure gradient at any given cross-section of the tube,

$$Q = -\pi\int_{r=0}^{r=h} r^2 f^{-1}\left\{\tfrac{1}{2}r\frac{\mathrm{d}P}{\mathrm{d}z}\right\}\mathrm{d}r. \tag{10}$$

For a given stress function $f\{\kappa\}$, one could, in principle, solve (10) for $\mathrm{d}P/\mathrm{d}z$. Substitution of (4) into the resulting expression and integration over one wavelength would then produce the pumping characteristic, $\Delta P_\lambda/\lambda$ versus $\bar{Q}$.

1.6. *Typical pumping performance*

The lubrication solution of Shapiro *et al.* (1969) exhibits several features common to all peristaltic pumping characteristics observed to date. It applies to a transverse sinusoidal wave of small slope transporting a Newtonian fluid of viscosity μ through an axisymmetric tube in the limit of zero Reynolds number, and takes the form

$$\frac{a^2}{4\mu c}\frac{\Delta P_\lambda}{\lambda} = \frac{\frac{1}{2}\phi^2(16-\phi^2)-(2+3\phi^2)\,\bar{Q}/\pi a^2 c}{(1-\phi^2)^{7/2}}, \tag{11}$$

where the tube radius is given by

$$h\{z\} = a + b\sin\{2\pi z/\lambda\} \tag{12}$$

in the wave frame, and $\phi \equiv b/a$ is called the *occlusion number* (or *amplitude ratio*). For finite a, $\phi = 0$ corresponds to no peristalsis, while $\phi = 1$ results in full occlusion, that

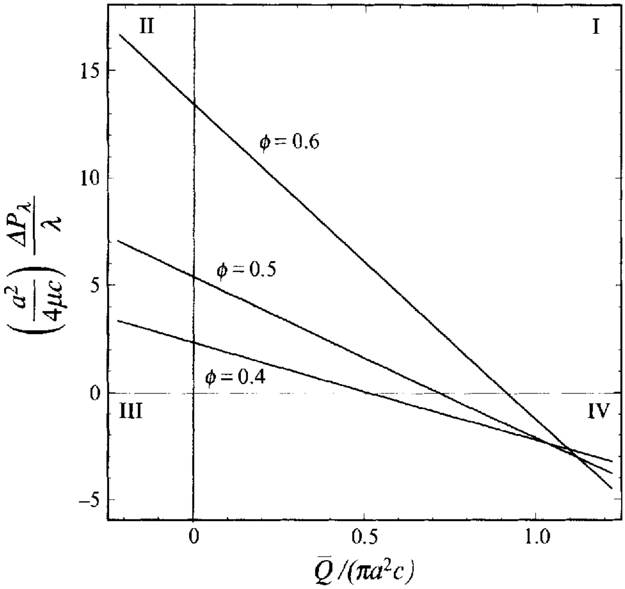

FIGURE 1. Pumping characteristics for a Newtonian fluid in the lubrication limit (Shapiro *et al.* 1969).

is, the tube is completely 'pinched off' at intervals of one wavelength. Pumping characteristics for several occlusions are shown in figure 1.

During free pumping ($\Delta P_\lambda = 0$), the mean flow is attributable solely to the motion of the peristaltic wave. When $\phi = 1$, fluid is trapped between points of complete occlusion, and the mean flow is in the direction of wave propagation ($\bar{Q} > 0$). If the occlusion is relaxed to some $\phi < 1$, the fluid is no longer completely trapped and 'leaks' backward through the contractions, resulting in a diminished mean flow rate. In the limit $\phi = 0$, the peristaltic wave vanishes, and $\bar{Q} = 0$. Setting $\Delta P_\lambda = 0$ in (11), we see that, during free pumping, the mean flow rate is positive for all occlusions and increases monotonically with ϕ (except in the trivial case $c = 0$). In fact, one can show that $\bar{Q}$ increases monotonically with ϕ even when $\Delta P_\lambda \neq 0$.

If the occlusion is fixed at some value $\phi < 1$, imposition of an adverse mean pressure gradient ($\Delta P_\lambda/\lambda > 0$) reduces the mean flow rate relative to the value obtained during free pumping. When $\Delta P_\lambda/\lambda$ is large enough, it balances exactly the driving force for flow produced by the wave, and the mean flow rate is zero. We emphasize that, according to (11), the value of the mean pressure gradient at which $\bar{Q} = 0$ is always positive (except in the trivial case $c = 0$).

The properties of the pumping characteristic (11) can be summarized as follows: (i) the mean flow rate increases monotonically with occlusion, and (ii) the pumping characteristic, $\Delta P_\lambda/\lambda$ versus $\bar{Q}$, has a negative slope and positive intercepts; it passes through the first quadrant of the $(\bar{Q}, \Delta P_\lambda/\lambda)$ plane.

As far as we are aware, the properties listed above hold true for all peristaltic pumping characteristics that have appeared in the literature to date. Even when the effects of wall slope, non-sinusoidal wave shapes, fluid inertia, and non-Newtonian fluid rheology have been taken into account, pumping characteristics have never been observed to pass through the third quadrant, and the mean flow rate has never been seen to decrease with increasing occlusion. In the following section we present an example of pumping performance that differs significantly from previously observed behaviour.

2. Pumping performance for a shear-thinning fluid

2.1. *Description of the model*

When the constitutive behaviour of shear-thinning (pseudoplastic) fluids in viscometric flows is examined over a wide range of shear rates, the apparent viscosity, defined by

$$\eta\{\kappa\} \equiv \tau_{rz}/\kappa = f\{\kappa\}/\kappa, \tag{13}$$

is often found to approach limiting values η_0 and η_∞ (where $\eta_\infty < \eta_0$) as $\kappa \to 0$ and $\kappa \to \infty$, respectively. The Reiner–Philippoff model,

$$\eta\{\kappa\} = \eta_\infty + (\eta_0 - \eta_\infty)[1+(\tau_{rz}/\tau_s)^2]^{-1}, \tag{14}$$

where τ_s is a third empirical parameter, exhibits this limiting behaviour and has been used to fit rheological data for a number of fluids. In the examples that follow, we use the nominal values $\eta_0 = 21.5$ cp, $\eta_\infty = 1.05$ cp, and $\tau_s = 0.073$ dyn cm^{-2}, which represent a fit to data taken for molten sulphur at 120 °C at shear stresses below 10 dyn cm^{-2} (Bird, Stewart & Lightfoot 1960) and are typical of shear-thinning fluids.

For a fluid described by (14), (10) reduces to

$$Q = -\left(\frac{\pi\tau_s h^3}{4\eta_0}\right)\left(\frac{\eta_r}{\alpha}\right)\left[\alpha^2 + 2(1-\eta_r)\left(1 - \frac{\ln\{1+\alpha^2/\eta_r\}}{\alpha^2/\eta_r}\right)\right], \tag{15}$$

where
$$\alpha \equiv \frac{h}{2\tau_s}\frac{\mathrm{d}P}{\mathrm{d}z}, \quad \eta_r \equiv \eta_0/\eta_\infty.$$

Given a wave shape, $h\{z\}$ (or $A\{z\}$), and speed, c, one can compute numerically the pumping characteristic, ΔP_λ versus $\bar{Q}$, as follows: (i) Choose a value of $\bar{Q}$; (ii) discretize the z-domain, then for each discrete value of z compute Q using (4) and solve (15) iteratively to get $\mathrm{d}P/\mathrm{d}z$; (iii) integrate $\mathrm{d}P/\mathrm{d}z$ numerically over one wavelength to obtain ΔP_λ. (It should be noted that since Q is an invertible function of $\mathrm{d}P/\mathrm{d}z$ (see proof of Lemma 4, Appendix, each value of Q corresponds to a unique value of $\mathrm{d}P/\mathrm{d}z$, for a given fluid and fixed A. Thus, the root computed in step (ii) of the numerical procedure is unique.)

The shape of the peristaltic wave used in the sample calculations is given by

$$h\{z\} = \begin{cases} a + b\cos\{2\pi z/\lambda_c\} & (0 \leqslant z \leqslant \lambda_c), \\ a+b & (\lambda_c < z \leqslant \lambda), \end{cases} \tag{16}$$

and is depicted in figure 2 with $a = 0.9$ cm, $b = 0.1$ cm, and $\lambda_c/\lambda = 0.2$. The axial dimension, z, appears scaled by the wavelength, λ, because, from the standpoint of lubrication theory, λ is arbitrary to the extent that it is large enough so that the effects of wall slope can be neglected.

2.2. *Effects of shear-dependent viscosity*

The solid curve in figure 3 represents the pumping characteristic for peristaltic transport of a Reiner–Philippoff fluid with $\eta_0 = 21.5$ cp, $\eta_\infty = 1.05$ cp, and $\tau_s = 0.073$ dyn/cm^2 by the waveform shown in figure 2. The dashed lines correspond to Newtonian fluids with viscosities of η_0 and η_∞. In all three cases the wave speed is 50 cm s^{-1}.

When the fluid is shear-thinning (solid curve), the pumping characteristic passes through the third quadrant of the $(\bar{Q}, \Delta P_\lambda/\lambda)$ plane; negative flow rates are achievable even for favourable mean pressure gradients. This is not the case when the fluid is

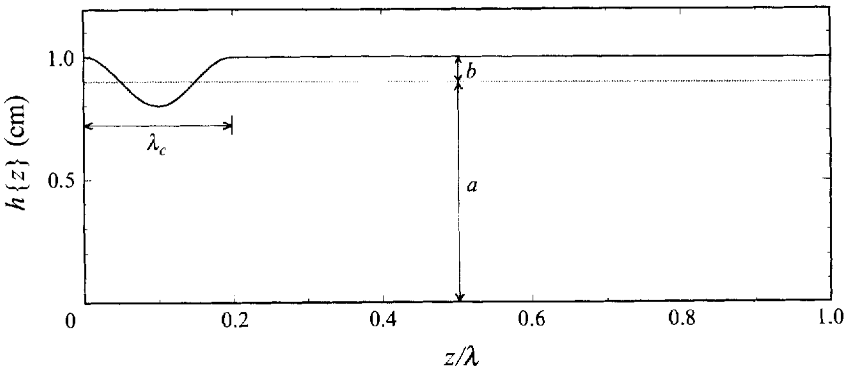

FIGURE 2. Waveform defined in (16), with $a = 0.9$ cm, $b = 0.1$ cm and $\lambda_c/\lambda = 0.2$.

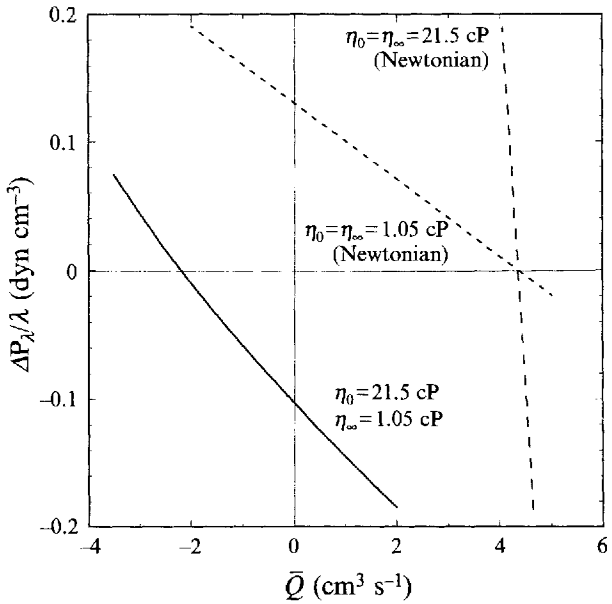

FIGURE 3. Pumping characteristics for the waveform in figure 2 transporting Newtonian and Reiner–Philippoff fluids. Note that negative free pumping occurs when the fluid is shear-thinning.

Newtonian. It is interesting to note that although the apparent viscosity of the Reiner–Philippoff fluid is bounded by the limiting values η_0 and η_∞, the solid curve in figure 3 is not bounded by the dashed lines. This illustrates the fact that negative free pumping is not simply the result of an 'overall' reduction in viscosity; the distribution of apparent viscosities throughout the tube is clearly of major importance.

2.3. *Dependence of mean flow on wave speed*

During free pumping, the peristaltic wave supplies the only driving force for fluid motion, and $\bar{Q} = 0$ when $c = 0$. As the wave speed is increased from zero, one might expect the mean flow rate to increase, as it does in figure 4 for Newtonian fluids (dashed line). This is, in fact, the case at values of c that are too small to be resolved in figure 4. However, in the previous section we demonstrated that, when the fluid is non-Newtonian, it is possible to have $\bar{Q} = 0$ when $c > 0$. In such a case, $\bar{Q}$ must decrease with increasing c over some range of wave speeds. In fact, $\bar{Q}$ can pass through a

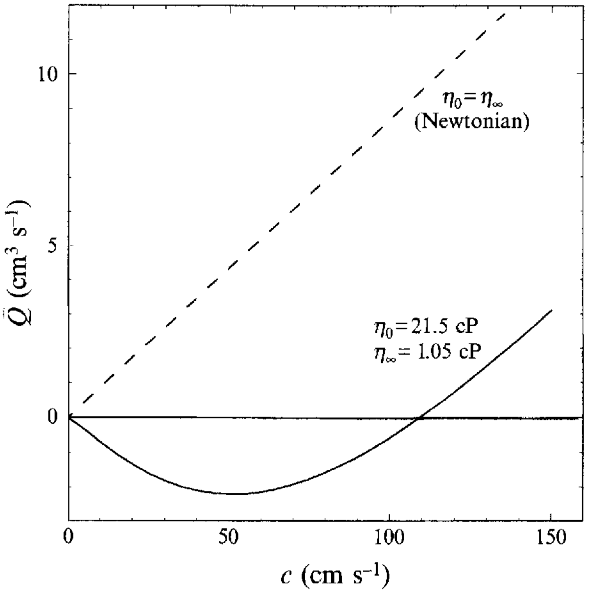

FIGURE 4. Variation of mean flow rate with wave speed for the waveform in figure 2 transporting Newtonian and Reiner–Philippoff fluids with $\Delta P_\lambda = 0$.

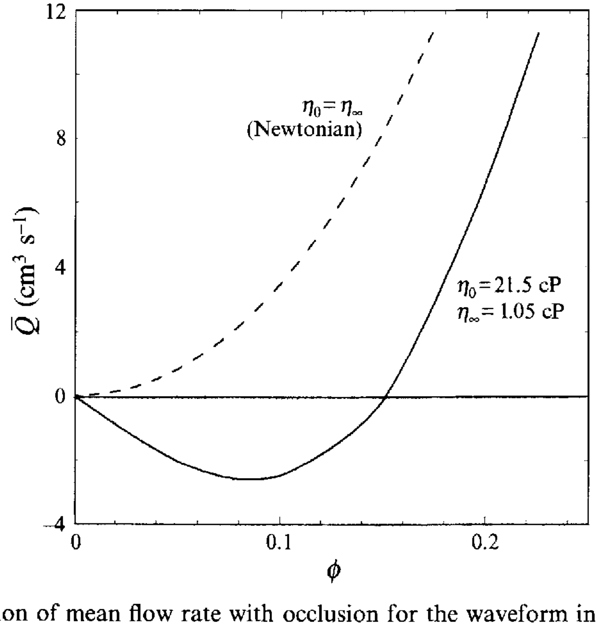

FIGURE 5. Variation of mean flow rate with occlusion for the waveform in figure 2 transporting Newtonian and Reiner–Philippoff fluids with $\Delta P_\lambda = 0$.

minimum value, as illustrated by the solid curve in figure 4. (As far as we know, Böhme & Friedrich 1983 are the only other workers to have observed a non-monotonic variation of the mean flow rate with wave speed. Their analysis of a small-amplitude wave pumping a second-order viscoelastic fluid uncovered a maximum in the mean flow rate, which they attributed to fluid memory effects.)

2.4. *Dependence of mean flow on occlusion*

The sinusoidal wave described by (12) has emerged as a prototype in the peristaltic literature. For this wave shape, the occlusion number, ϕ, is defined as the ratio of the maximum wave amplitude, b, to the average tube radius, a. Extension of this definition to other waveforms can proceed in a number of ways. For the present purposes, $\phi \equiv b/a$ should suffice as a measure of the degree to which the composite wave defined in (16) is occluded. The parameters a and b will be adjusted such that the volume enclosed by the wave remains constant as ϕ is varied.

Consider again the case of free pumping. When $\phi = 0$, the tube does not deform, and $\bar{Q} = 0$. When $\phi = 1$, the tube is completely closed off at intervals of one wavelength, and $\bar{Q} > 0$. Figure 5 shows the dependence of $\bar{Q}$ on ϕ in the range of low occlusion. For Newtonian fluids (dashed curve), the mean flow rate increases steadily with occlusion, as expected (§1.6). However, in the case of the shear-thinning fluid (solid curve), the mean flow rate immediately becomes negative when the occlusion is first increased from zero. As ϕ is increased further, $\bar{Q}$ reaches a negative minimum value, then increases steadily towards its positive limiting value at full occlusion.

3. Mathematical treatment of free pumping

3.1. *The straight-section-dominated (SSD) wave model*

Before moving on to general arguments concerning the effects of wave shape and fluid properties on pumping performance, we pause here to describe the basic viscous mechanism responsible for peristaltic pumping and to introduce a special type of wave that will be useful in the subsequent analysis. To construct such a wave, one begins with two sections of tubing, one of length λ_{T1}, the other of length λ_{T2}, and each of radius h_1 at one end and h_2 at the other end, where $h_1 < h_2$. The changes in radius should occur gradually enough so that the lubrication approximation can be applied to flow through these sections. One then connects the narrow ends of these two sections with a long, straight section of radius h_1 and length, λ_1, much greater than λ_{T1} and λ_{T2}. Next, one appends to one of the free ends a long, straight section of radius h_2 and length, λ_2, also much greater than λ_{T1} and λ_{T2}. Finally, this assembly of four sections, shown in figure 6, is used as the repeating unit in an infinite train of peristaltic waves. This waveform is related closely to the 'sliding cuff' model introduced by Shapiro *et al.* (1969) and later used by Becker (1980).

If λ_1 and λ_2 are large enough compared with λ_{T1} and λ_{T2}, the contributions of the short, transition sections to the mean flow and mean pressure gradient become negligible compared to the contributions from the main, straight sections (see Lemma 4, Appendix). Thus, for the purpose of determining pumping characteristics, only the straight sections are important. Accordingly, we will refer to such waves as straight-section-dominated (SSD) waves. It must be stressed that SSD waves are constructed such that changes in radius occur gradually over the length of each transition section, so that lubrication theory is applicable.

This simple SSD wave can be used to gain insight into the physics of free pumping ($\Delta P_\lambda = 0$). Let the contracted and expanded straight sections have cross-sectional areas A_1 and A_2, respectively, where $A_2 > A_1$. Figure 6 depicts such a wave translating from left to right. We will refer to the transition sections as contracting or expanding transitions, depending on whether they bring about a decrease or an increase in cross-sectional area, respectively, as they pass an observer fixed in the laboratory frame.

Consider a cylindrical volume element of differential length, dZ, whose axial

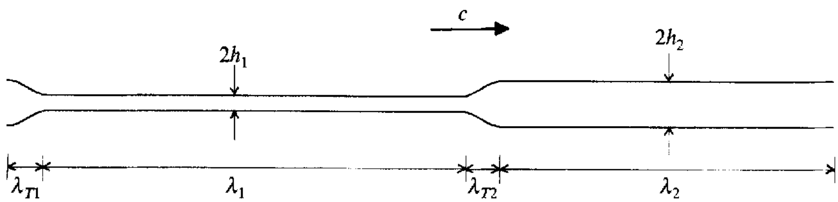

FIGURE 6. Sketch used to define a straight-section-dominated (SSD) wave.

Sign of Q_1	Sign of Q_2	Relative magnitudes
$(-)$	$(-)$	$\lvert Q_1\rvert > \lvert Q_2\rvert$
$(+)$	$(+)$	$\lvert Q_1\rvert < \lvert Q_2\rvert$
$(-)$	$(+)$	$\lvert Q_1\rvert > \lvert Q_2\rvert$
$(-)$	$(+)$	$\lvert Q_1\rvert < \lvert Q_2\rvert$

TABLE 1. Signs and relative magnitudes of Q_1 and Q_2 consistent with the influxes and outflows associated with contracting and expanding transitions

position is fixed in the laboratory frame and whose radius corresponds to the tube radius. As a contracting transition passes by, the volume of the element, $A\{Z\}\,\mathrm{d}Z$, decreases. For an incompressible fluid, mass conservation demands that the reduction in volume be accompanied by a net flow of fluid out of the element. Thus, a net efflux of fluid is associated with a contracting transition. Similarly, a net influx of fluid is associated with an expanding transition. Let Q_1 and Q_2 denote the flow rates through the contracted and expanded (straight) sections, respectively. Table 1 shows the possible ways in which the signs and magnitudes of Q_1 and Q_2 can combine to create the required influxes and outflows.

The net pressure change over one wavelength is given by

$$\Delta P_\lambda = \lambda_1 \left.\frac{\mathrm{d}P}{\mathrm{d}z}\right|_1 + \lambda_2 \left.\frac{\mathrm{d}P}{\mathrm{d}z}\right|_2, \tag{17}$$

where λ_1 and λ_2 are the lengths of the contracted and expanded sections, respectively. In the case of free pumping, (17) implies that the pressure gradients in the contracted and expanded sections must be of opposite sign (except in the trivial case where $\mathrm{d}P/\mathrm{d}z = 0$ in both sections). Equation (10), together with the assumption that the stress function takes the same sign as its argument (§1.4), then implies that the flow rates Q_1 and Q_2 induced by these pressure gradients are also of opposite sign. Table 1 indicates that $Q_1 < 0$ and $Q_2 > 0$.

The mean flow rate is

$$\bar{Q} = (\lambda_1 Q_1 + \lambda_2 Q_2)/\lambda, \tag{18}$$

where $\lambda = \lambda_1 + \lambda_2$ is the total wavelength. With $\Delta P_\lambda = 0$, (17) and (18) can be combined to yield

$$\bar{Q} = \frac{\lambda_2 Q_2}{\lambda}\left(1 - \frac{\Omega_2}{\Omega_1}\right), \tag{19}$$

where

$$\Omega_i \equiv \frac{\left.\dfrac{\mathrm{d}P}{\mathrm{d}z}\right|_i}{-Q_i}. \tag{20}$$

The quantity Ω can be considered a resistance to fluid flow per unit length of tube.

Since λ, λ_2, and Q_2 are all positive, (19) implies that the sense of the mean flow depends on whether Ω_2 is greater than, equal to, or less than Ω_1. Specifically, $\Omega_2 > \Omega_1$, $\Omega_2 = \Omega_1$, and $\Omega_2 < \Omega_1$ correspond to $\bar{Q} < 0$, $\bar{Q} = 0$, and $\bar{Q} > 0$, respectively.

The following picture of free pumping by an SSD wave emerges. The expanding and contracting transitions draw in and expel fluid, respectively, inducing negative flow in the contracted sections and positive flow in the expanded sections of the tube. The mean flow rate is a weighted sum of these positive and negative flows. Its sign depends on which straight sections (contracted or expanded) pose the lesser resistance to flow per unit length of tube.

For a Newtonian fluid of viscosity μ, (10) reduces to Poiseuille's law,

$$Q = -\frac{A^2}{8\pi\mu}\frac{\mathrm{d}P}{\mathrm{d}z}. \tag{21}$$

Comparison of (21) and (20) shows that $\Omega_i = 8\pi\mu/A_i^2$. Thus, for a given viscosity, Ω_i is always larger for a smaller cross-section than for a larger cross-section. It follows from (19) that, for a Newtonian fluid driven by an SSD wave, free pumping always results in positive mean flow.

The remainder of the paper is devoted to exploring the conditions under which peristalsis fails to generate a positive mean flow despite the absence of an adverse mean pressure gradient. In the next section, we formalize the arguments presented above and generalize them to allow for arbitrary wave shapes.

3.2. *A theorem regarding the direction of mean flow*

The analysis that follows is concerned with non-trivial waveforms, for which $A\{z\} \not\equiv$ constant, and $A\{z\} > 0$ for all z. Intermediate results of major importance appear in the main text as Lemmas 1–3. These are supplemented by Lemmas 4–10 in the Appendix.

3.2.1. *Properties of Ω*

The arguments of the previous section highlighted the importance of the resistance to flow per unit length of tube, Ω, defined by

$$Q = -\frac{1}{\Omega}\frac{\mathrm{d}P}{\mathrm{d}z}, \tag{22}$$

in determining the sense of the mean flow. Throughout the remainder of this paper, we will refer to Ω simply as the resistance to flow.

Equation (10) implies that, for a given fluid, $\mathrm{d}P/\mathrm{d}z$ is a function only of Q and A. By (22), Ω must also be a function only of Q and A. However, (4) indicates that $Q = Q\{A\{z\}; \bar{A}, c, \bar{Q}\}$. Therefore, we can write $\Omega = \Omega\{A, Q\} = \Omega\{A\{z\}; \bar{A}, c, \bar{Q}\}$. Since $\bar{A}$, c, and $\bar{Q}$ do not vary with position and time, the z-dependence of Ω enters through $A\{z\}$ alone. In the analysis that follows, we will often abbreviate the functional form of Ω to $\Omega\{A\{z\}\}$ to focus attention on the z-dependence.

3.2.2. *Pumping characteristic in terms of Ω*

Equating the right-hand sides of (4) and (22), solving for $\mathrm{d}P/\mathrm{d}z$, and integrating over one wavelength, we get

$$\Delta P_\lambda = -\int_0^\lambda [\bar{Q} + c(A - \bar{A})]\,\Omega\,\mathrm{d}z. \tag{23}$$

Rearrangement of (23) yields an equation for the time-mean flow rate in terms of the resistance to flow:

$$\bar{Q} = \frac{-c}{\lambda\bar{\Omega}}\int_0^\lambda [A\{z\}-\bar{A}]\,\Omega\{A\{z\};\bar{A},c,\bar{Q}\}\,\mathrm{d}z - \frac{\Delta P_\lambda}{\lambda\bar{\Omega}}, \tag{24}$$

where $\bar{\Omega} = \bar{\Omega}\{\bar{A},c,\bar{Q}\}$.

3.2.3. *Resistance inversion in negative free pumping*

For negative free pumping to be possible, (24) with $\Delta P_\lambda = 0$ must have a solution $\bar{Q} < 0$ for some choice of $A\{z\}$ and c. Assume that such a solution exists. Since Ω is always positive (Lemma 5), $\bar{\Omega}$ is positive. It follows from (24) that

$$\int_0^\lambda [A\{z\}-\bar{A}]\,\Omega\{A\{z\}\}\,\mathrm{d}z > 0. \tag{25}$$

Let Ω_1^{min} be the minimum value of Ω over all z for which $A\{z\} < \bar{A}$. Suppose there does not exist a value of z for which $A\{z\} > \bar{A}$ and $\Omega\{A\{z\}\} > \Omega_1^{min}$. Then $[A\{z\}-\bar{A}]\,\Omega\{A\{z\}\} \leqslant [A\{z\}-\bar{A}]\,\Omega_1^{min}$ for all z, and

$$\int_0^\lambda [A\{z\}-\bar{A}]\,\Omega\{A\{z\}\}\,\mathrm{d}z \leqslant \Omega_1^{min}\int_0^\lambda [A\{z\}-\bar{A}]\,\mathrm{d}z. \tag{26}$$

However, by the definition of $\bar{A}$,

$$\int_0^\lambda [A\{z\}-\bar{A}]\,\mathrm{d}z = 0. \tag{27}$$

The result of substituting (27) into (26) contradicts condition (25). Therefore, for (25) to be satisfied, there must exist cross-sections of area $A_1 < \bar{A}$ and $A_2 > \bar{A}$ such that $\Omega\{A_2\} > \Omega\{A_1\}$. As foreshadowed in §3.1, negative free pumping requires an 'inversion' of resistances; some cross-section must pose less resistance to flow than some larger cross-section. This result implies the following:

LEMMA 1. *With $\Delta P_\lambda = 0$, if $\Omega\{A_2, Q_2\} < \Omega\{A_1, Q_1\}$ for all $A_1 < A_2$, Q_1, and Q_2, then $\bar{Q} \geqslant 0$ for all waveforms.*

3.2.4. *Resistance equality in zero free pumping*

With $\Delta P_\lambda = 0$ and $\bar{Q} = 0$, (24) reduces to

$$\int_0^\lambda [A\{z\}-\bar{A}]\,\Omega\{A\{z\}\}\,\mathrm{d}z = 0. \tag{28}$$

Following logic analogous to that used to derive Lemma 1, we conclude that if there does not exist a z for which $A\{z\} > \bar{A}$ and $\Omega\{A\{z\}\} \geqslant \Omega_1^{min}$, then

$$\int_0^\lambda [A\{z\}-\bar{A}]\,\Omega\{A\{z\}\}\,\mathrm{d}z < 0. \tag{29}$$

Similarly, if there does not exist a z for which $A\{z\} > \bar{A}$ and $\Omega\{A\{z\}\} \leqslant \Omega_1^{max}$, where Ω_1^{max} is the maximum value of Ω over all z for which $A\{z\} < \bar{A}$, then

$$\int_0^\lambda [A\{z\}-\bar{A}]\,\Omega\{A\{z\}\}\,\mathrm{d}z > 0. \tag{30}$$

Therefore, if (28) is to be satisfied, there must exist at least one cross-section of area $A_2 > \bar{A}$ such that $\Omega\{A_2\} \geqslant \Omega_1^{min}$, and at least one of area $A_2 > \bar{A}$ such that $\Omega\{A_2\} \leqslant \Omega_1^{max}$. By Lemma 6, Ω is a continuous function of A. Since $A\{z\}$ is also continuous, it follows that $\Omega\{A_2\} = \Omega\{A_1\}$ for some $A_2 > A_1$. Thus, equality of resistances for cross-sections of different areas is a necessary condition for free pumping to result in $\bar{Q} = 0$ for some waveform. Next, we show that it is also a sufficient condition.

Recall that for a given fluid $\Omega = \Omega\{A, Q\}$, and suppose that $\Omega\{A_2, Q_2\} = \Omega\{A_1, Q_1\}$ for some $A_1 < A_2$, Q_1, and Q_2. (Since, by Lemma 7, $\Omega\{A, -Q\} = \Omega\{A, Q\}$, we may assume without loss of generality that $Q_1 < 0$ and $Q_2 > 0$.) This supposition implies that Q_1 and Q_2 are permissible flow rates for areas A_1 and A_2, respectively, that is, the inherent limitations of the model are not exceeded by these flows. It follows from Lemma 8 that, with $\Delta P_\lambda = 0$, one can construct an SSD wave for which the resistances to flow in the contracted and expanded sections are equal. Furthermore, the arguments of §3.1 show that for an SSD wave $\Omega_2 = \Omega_1$ implies $\bar{Q} = 0$. Thus, if $\Omega\{A_2, Q_2\} = \Omega\{A_1, Q_1\}$ for some $A_1 < A_2$, Q_1, and Q_2, then there exists a waveform (at the very least, an SSD wave) for which $\bar{Q} = 0$.

Combining the results of this subsection, we find that having $\Omega\{A_2, Q_2\} = \Omega\{A_1, Q_1\}$ for some $A_1 < A_2$, Q_1, and Q_2 is necessary and sufficient for free pumping to result in zero net flow for at least one wave motion, described by $A\{z\}$ and c. Equivalently, $\bar{Q} \neq 0$ for all waveforms if and only if $\Omega\{A_2, Q_2\} \neq \Omega\{A_1, Q_1\}$ for all $A_1 < A_2$, Q_1, and Q_2. With the help of Lemma 9, we can write

LEMMA 2. *With* $\Delta P_\lambda = 0$, $\bar{Q} \neq 0$ *for all waveforms if and only if* Ω *depends only on* A.

3.2.5. *Resistance inequality and positive free pumping*

If $\bar{Q} > 0$ for all waveforms, it follows trivially that $\bar{Q} \neq 0$, which, by Lemma 2, implies that Ω depends only on A. Conversely, if Ω depends only on A, then: (i) by Lemma 2, $\bar{Q} \neq 0$, and (ii) by the Corollary to Lemma 9, $\Omega\{A_2, Q_2\} < \Omega\{A_1, Q_1\}$ for all $A_1 < A_2$, Q_1, and Q_2, which, according to Lemma 1, implies that $\bar{Q} \geqslant 0$, and so (iii) $\bar{Q} > 0$ for all waveforms. We have just proven the following:

LEMMA 3. *With* $\Delta P_\lambda = 0$, $\bar{Q} > 0$ *for all waveforms if and only if* Ω *depends only on* A.

3.2.6. *Constitutive behaviour and positive free pumping*

Lemma 3 provides a condition, expressed in terms of Ω, that is necessary and sufficient to ensure that all (non-trivial) waveforms give rise to positive pumping. To make this result more useful, we relate the condition on Ω to a condition on the constitutive behaviour of the transported fluid, which is characterized by the stress function $f\{\kappa\}$ defined in (5).

Dividing (10) through by $-\mathrm{d}P/\mathrm{d}z$ and recalling the definition of Ω, we get

$$K\{h, \gamma\} \equiv -\pi \int_{r=0}^{r=h} r^2 f^{-1}\{\tfrac{1}{2} r\gamma\}/\gamma \,\mathrm{d}r, \tag{31}$$

where $K \equiv 1/\Omega$, $\gamma \equiv \mathrm{d}P/\mathrm{d}z$, and, as before, $h \equiv (A/\pi)^{1/2}$. Clearly, Ω depends on A alone if and only if K depends on h alone. Suppose that $K = K\{h\}$. Differentiation of (32) gives $K'\{h\} = h^2 f^{-1}\{\tfrac{1}{2} h\gamma\}/\gamma$, which implies directly that $f^{-1}\{\tfrac{1}{2} r\gamma\}/\gamma$ depends only on r. Conversely, if $f^{-1}\{\tfrac{1}{2} r\gamma\}/\gamma$ depends only on r, then it follows trivially from (32) that K depends on h alone. We conclude that Ω depends on A alone if and only if $f^{-1}\{\tfrac{1}{2} r\gamma\}/\gamma$ depends only on r, which, in turn, is true if and only if the fluid is Newtonian (Lemma 10). Combination of this result with Lemma 3 leads to our main conclusion:

FREE PUMPING THEOREM (FPT). *With $\Delta P_\lambda = 0$, $\bar{Q} > 0$ for all waveforms if and only if the transported fluid is Newtonian.*

4. Discussion

4.1. *Significance and limitations of the free pumping theorem*

According to the FPT, free pumping of a Newtonian fluid under the conditions of §1.4 results in positive mean flow for all (non-trivial) waveforms. The same conclusion can be reached using Lemma 3. Recall from §3.1 that $\Omega = 8\pi\mu/A^2$ for a Newtonian fluid of viscosity μ. Clearly, Ω depends solely on A, which implies that $\bar{Q} > 0$ for all waveforms.

Perhaps the more significant aspect of the FPT is its assertion that when $\Delta P_\lambda = 0$, $\bar{Q} > 0$ for all waveforms only if the fluid is Newtonian. In other words, any shear-thinning or shear-thickening behaviour embodied in the stress function $f\{\kappa\}$ would allow $\bar{Q} \leqslant 0$ for at least one waveform.

In §3.2 we introduced the idea of 'permissible' flow rates in recognition of the fact that the range of validity of our peristaltic model is necessarily limited by its underlying assumptions and by the finite range of shear stresses over which the constitutive behaviour of the fluid is characterized. Accordingly, the set of 'all waveforms' referred to in the FPT must be restricted to include only those waveforms that do not result in a breach of the inherent limitations of the lubrication analysis. Similarly, non-Newtonian fluid rheology leads to non-positive pumping for at least one waveform only if this is not precluded by a breakdown of the model.

4.2. *Mechanism for 'inverting' resistances*

In §3.1 we showed that for the simple SSD wave, the sense of the mean flow during free pumping depends on the ratio of the resistances to flow in the contracted and expanded sections, Ω_1 and Ω_2, respectively. In particular, negative free pumping results when $\Omega_2 > \Omega_1$, which never occurs if the transported fluid is Newtonian. In the discussion that follows, we explain how such an inversion of resistances can occur when the viscosity of the fluid is shear-dependent. We use the example in §2.1 to demonstrate that the conclusions drawn from our analysis of SSD waves can provide a qualitative understanding of peristaltic transport by other waveforms.

Although the waveform defined by (16) and shown in figure 2 is not an SSD wave, it conforms to the assumption of small wall slope when λ is sufficiently large, and can be thought of as having a section that is, on average, 'contracted'. The remainder of the wavelength is considered to be 'expanded'. For the purpose of this discussion, we define the contracted section to be the interval in which $A\{z\} < \bar{A}$. For the wave in figure 2, this corresponds approximately to $0.020 < z/\lambda < 0.180$.

Let us focus our attention on the intersection of the solid curve with the horizontal axis in figure 3. This point corresponds to negative free pumping of a shear-thinning fluid by the waveform in figure 2. Figure 7 shows the axial variation of A, Q, $\langle\eta\rangle$, and Ω, where the angle brackets denote a cross-sectional average. Open and solid circles indicate the average values of these quantities in the contracted and expanded sections, respectively. (The peak values of $\langle\eta\rangle$ and Ω, which are off-scale in figures 7(*c*) and 7(*d*), are finite and correspond to $Q = 0$.)

In the contracted section, the average cross-sectional area is, of course, less than in the expanded section. For a Newtonian fluid, this would result in a higher average resistance in the contracted section than in the expanded section. However, with the shear-thinning fluid, the average resistance in the contracted section falls below that in

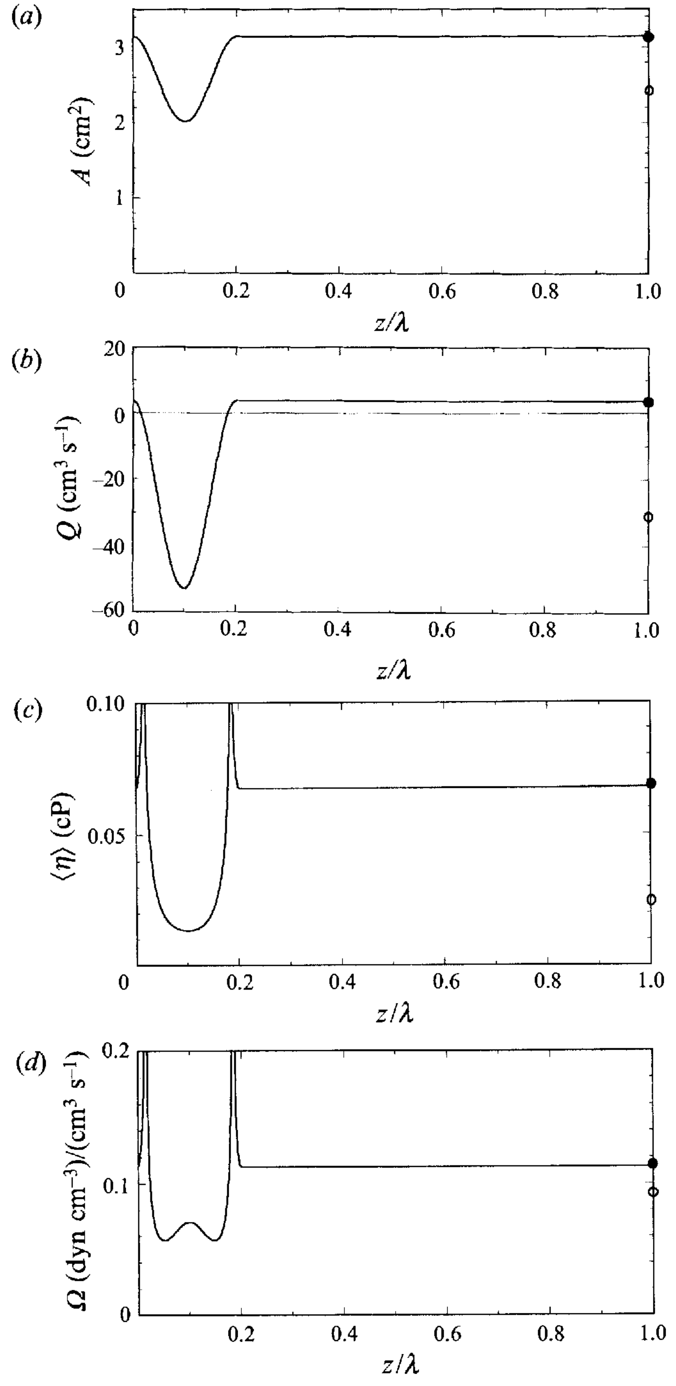

FIGURE 7. Axial profiles of (*a*) area, A, (*b*) flow rate, Q, (*c*) cross-sectionally averaged η, and (*d*) resistance, Ω, during negative free pumping. Circles, ○ and ●, denote averages in the contracted and expanded sections, respectively.

the expanded section, as shown in figure 7(*d*). This can occur because the resistance is a function of both A and Q. The flow through a given cross-section is dominated by successively higher shear rates as the magnitude of Q is increased, resulting in successively lower apparent viscosities, which in turn decreases the resistance to flow. Thus, a low resistance can be achieved in a small cross-section by inducing a large flow rate (either positive or negative). This is, in fact, what occurs in this example. The large, negative flow rates in the contracted section (figure 7*b*) produce values of $\langle \eta \rangle$ that approach $\eta_{\infty} = 1.05$ cp (figure 7*c*). The shear-thinning effect is sufficient to bring about an inversion of average resistances between the contracted and expanded sections (figure 7*d*).

4.3. *Importance of wave shape*

To our knowledge, the present work is the first to identify negative mean flow in the presence of a zero or favourable mean pressure gradient. We believe that this effect has been overlooked in previous work primarily because of the wave shapes required to bring it about. Ironically, for a shear-thinning fluid, these shapes resemble those found in commercial roller pumps in that the sections of reduced cross-sectional area are short compared to the total wavelength.

For the sake of simplicity, consider an SSD wave. For a shear-thinning fluid, equality of resistances is induced by having a large flow rate (high shear) in the contracted sections compared to that in the expanded sections, as discussed in §4.2. Now, zero mean flow implies that $\lambda_1 Q_1 + \lambda_2 Q_2 = 0$. If $Q_1 < 0$ is to be much larger in magnitude than $Q_2 > 0$, then $\lambda_1 \ll \lambda_2$; the contracted sections must be significantly shorter than the expanded sections. (One would expect the situation for negative mean flow to be even more extreme.) The waveform used to demonstrate negative free pumping in §2, though not an SSD wave, conforms qualitatively to this requirement. For a shear-thickening fluid, analogous reasoning suggests that the contracted sections must be significantly longer than the expanded sections. Thus, it is likely that previous investigators did not consider waves of sufficiently skewed shape to induce zero or negative free pumping. Interestingly, Becker (1980) computed characteristics for transport of a Prandtl–Eyring fluid by an SSD wave in a regime that yields negative free pumping. (Although the analysis was performed for a single wave in a tube of finite length, the results are applicable to one wavelength of an infinite train of waves.) However, the author was interested only in the flow induced in the uncontracted sections of the pipe, Q_2, which is necessarily positive when $\Delta P_{\lambda} = 0$ (§3.1), and did not compute the mean flow rate, $\bar{Q}$.

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Appendix

In this section we sketch derivations of several ancillary results referred to in the main text.

LEMMA 4. *Consider the waveform in figure 6. If λ_1 and λ_2 are large enough compared with λ_{T1} and λ_{T2}, the contributions of the short, transition sections to the pumping characteristic are negligible.*

For the waveform pictured in figure 6, the mean flow rate is

$$\bar{Q} = \frac{1}{\lambda}\left(\lambda_1 Q_1 + \lambda_2 Q_2 + \int_T Q \, \mathrm{d}z\right), \tag{A 1}$$

where T indicates that the integration is to be performed over both transition sections, whose combined length is $\lambda_T \equiv \lambda_{T1} + \lambda_{T2}$. Now, application of (4) to two points, z and z_1, and elimination of $\bar{Q}$ yields

$$Q = Q_1 + c(A - A_1), \tag{A 2}$$

where quantities evaluated at z are written without subscripts, and z_1 is in straight section 1. Substitution of (A 2) into the integral term in (A 1), and recognition that $A \geqslant A_m$, where A_m is the minimum cross-sectional area, gives

$$\int_T Q \,\mathrm{d}z \geqslant \lambda_T[Q_1 + c(A_m - A_1)]. \tag{A 3}$$

Similarly, one can show that

$$\int_T Q \,\mathrm{d}z \leqslant \lambda_T[Q_2 + c(A_M - A_2)], \tag{A 4}$$

where A_M is the maximum cross-sectional area (assumed to be finite). Using (A 3) and (A 4) in (A 1), we obtain

$$\bar{Q}^{SSD} + \frac{\lambda_T}{\lambda}[Q_1 + c(A_m - A_1)] \leqslant \bar{Q} \leqslant \bar{Q}^{SSD} + \frac{\lambda_T}{\lambda}[Q_2 + c(A_M - A_2)], \tag{A 5}$$

where
$$\bar{Q}^{SSD} \equiv \frac{1}{\lambda}(\lambda_1 Q_1 + \lambda_2 Q_2). \tag{A 6}$$

Given the assumptions made in § 1.4 regarding the stress function, f, one can deduce from (10) that Q is a continuously differentiable, strictly decreasing (and therefore invertible) function of $\gamma \equiv \mathrm{d}P/\mathrm{d}z$, and that $\partial Q/\partial \gamma$ is bounded. Using this information, one can employ the same reasoning as above to show that

$$\left(\frac{\Delta P_\lambda}{\lambda}\right)^{SSD} + \left(\frac{\lambda_T}{\lambda}\right)\gamma\{Q_1 + c(A_m - A_1)\} \leqslant \frac{\Delta P_\lambda}{\lambda} \leqslant \left(\frac{\Delta P_\lambda}{\lambda}\right)^{SSD} + \left(\frac{\lambda_T}{\lambda}\right)\gamma\{Q_2 + c(A_M - A_2)\}, \tag{A 7}$$

where
$$\left(\frac{\Delta P_\lambda}{\lambda}\right)^{SSD} \equiv \frac{1}{\lambda}(\lambda_1 \gamma_1 + \lambda_2 \gamma_2). \tag{A 8}$$

Now, if one increases λ_1 and λ_2 so that $\lambda_1 \gg \lambda_T$ and $\lambda_2 \gg \lambda_T$, it follows from (A 5) and (A 7) that the mean flow rate and mean pressure gradient are dominated by the contributions from the straight sections, provided Q_1 and Q_2 do not go to zero during this limiting process. (If Q_1 and Q_2 were allowed to vanish in (A 5)–(A 8), it can be shown that one would be left with $\bar{Q} = 0$ and $\Delta P_\lambda/\lambda = 0$. However, since one is always free to specify either $\bar{Q}$ or $\Delta P_\lambda/\lambda$ when posing the pumping problem, this restriction is physically unrealistic. Therefore, Q_1 and Q_2 do not vanish.)

LEMMA 5. *Ω is always positive.*

Recall that the function $f\{\kappa\}$ in (5) takes the same sign as its argument κ. Clearly, then, $f^{-1}(\xi\}$ also takes the same sign as its argument ξ. Therefore, the sign of the integrand in (10) is the same as that of $\mathrm{d}P/\mathrm{d}z$ (except possibly at $r = 0$), and is constant throughout the entire range of integration. It follows that Q and $\mathrm{d}P/\mathrm{d}z$ are of opposite sign, and $\Omega > 0$.

LEMMA 6. *For fixed $\mathrm{d}P/\mathrm{d}z$, Ω is a continuous, strictly decreasing function of A.*

Since the limits of integration in (10) are continuous in A, Q can vary discontinuously with A only if the integrand (specifically, the function f^{-1}) contains a singularity. Since, by assumption, no such singularity exists (§ 1.4), Q is continuous in A for fixed $\mathrm{d}P/\mathrm{d}z$. It follows directly that Ω is continuous in A for fixed $\mathrm{d}P/\mathrm{d}z$.

Now, let $\gamma \equiv \mathrm{d}P/\mathrm{d}z$. Differentiation of (10) with respect to A gives

$$\left.\frac{\partial Q}{\partial A}\right|_\gamma = \left.\frac{\partial Q}{\partial h}\right|_\gamma \frac{\mathrm{d}h}{\mathrm{d}A} = -\tfrac{1}{2}hf^{-1}\{\tfrac{1}{2}h\gamma\}. \tag{A 9}$$

Then

$$\left.\frac{\partial \Omega}{\partial A}\right|_\gamma = \left.\frac{\partial \Omega}{\partial Q}\right|_\gamma \left.\frac{\partial Q}{\partial A}\right|_\gamma = -\frac{h\gamma}{2Q^2}f^{-1}\{\tfrac{1}{2}h\gamma\}. \tag{A 10}$$

Recall that $f^{-1}\{\xi\}$ takes the same sign as its argument. Assuming $h > 0$, it follows from (A 10) that $(\partial\Omega/\partial A)_\gamma < 0$. Thus, Ω is a strictly decreasing function of A.

LEMMA 7. *For fixed A, Ω is an even function of Q.*

Since $f\{\kappa\}$ is an odd function of its argument, so is $f^{-1}\{\xi\}$. Thus, changing the sign of $\mathrm{d}P/\mathrm{d}z$ simply changes the sign, but not the magnitude, of the integrand in (10). As a result, changing the sign of $\mathrm{d}P/\mathrm{d}z$ changes only the sign of Q. It follows from the definition of Ω that $\Omega\{A, -Q\} = \Omega\{A, Q\}$.

LEMMA 8. *Let $Q_1 < 0$ and $Q_2 > 0$ be permissible flow rates for cross-sectional areas A_1 and A_2, respectively. One can construct an SSD wave with contracted area A_1 and expanded area A_2 such that during free pumping Q_1 and Q_2 are realized in the contracted and expanded sections of the tube, respectively.*

Let $\gamma \equiv \mathrm{d}P/\mathrm{d}z$. As mentioned in the proof of Lemma 4, Q is an invertible function of γ, and $\partial Q/\partial\gamma$ is continuous, negative, and bounded. Thus, for fixed A, we may write $\gamma = g\{Q\}$, where $\mathrm{d}g/\mathrm{d}Q$ is continuous, negative, and bounded. For a given SSD wave, with $\Delta P_\lambda = 0$, (17) then leads to

$$\Lambda g\{Q_1\} + g\{Q_2\} = 0, \tag{A 11}$$

where $\Lambda \equiv \lambda_1/\lambda_2 > 0$. Writing (1) for the contracted and expanded sections of the tube and eliminating q, we obtain

$$-Q_1 + Q_2 = c(A_2 - A_1), \tag{A 12}$$

where $A_2 > A_1$ by convention. Equations (A 11) and (A 12) define a transformation from the space of points (Q_1, Q_2) to the space of points (c, Λ). One can show that, for all non-trivial SSD waves, the Jacobian of this transformation does not vanish. Thus, each point (Q_1, Q_2) is associated with a unique pair (c, Λ). (Furthermore, if $Q_1 < 0$ and $Q_2 > 0$, then $c > 0$ and $\Lambda > 0$.) This implies that we may induce any permissible values of $Q_1 < 0$ and $Q_2 > 0$ by choosing c and λ_1/λ_2 appropriately. It follows that any flow rates $Q_1 < 0$ and $Q_2 > 0$ that are permissible for cross-sectional areas A_1 and A_2 can be realized in the contracted and expanded sections of an SSD wave with contracted and expanded areas A_1 and A_2.

LEMMA 9. *$\Omega\{A_2, Q_2\} \neq \Omega\{A_1, Q_1\}$ for all $A_1 < A_2$, Q_1, and Q_2 if and only if Ω depends on A alone.*

COROLLARY. *If Ω depends on A alone, then $\Omega\{A_2, Q_2\} < \Omega\{A_1, Q_1\}$ for all $A_1 < A_2$, Q_1, and Q_2.*

Assume $\Omega\{A_2, Q_2\} \neq \Omega\{A_1, Q_1\}$ for all $A_1 < A_2$, Q_1, and Q_2. Since A_1 and A_2 are interchangeable. $\Omega\{A_2, Q_2\} \neq \Omega\{A_1, Q_1\}$ for all $A_1 \neq A_2$, Q_1, and Q_2. Thus, all points (A, Q) on the locus of points that satisfy $\Omega = \text{constant}$ share the same value of A. It follows that each value of Ω is associated with a single value of A; Ω depends only on A.

Conversely, assume Ω depends on A alone. Since Ω is a strictly decreasing function of A (Lemma 6), if $A_1 < A_2$, then $\Omega\{A_2\} < \Omega\{A_1\}$.

LEMMA 10. *$f^{-1}\{\frac{1}{2}r\gamma\}/\gamma$ depends on r alone if and only if $f\{\kappa\} = \mu\kappa$, where μ is a constant.*

Assume that $f^{-1}\{\frac{1}{2}r\gamma\}/\gamma = F\{r\}$. It follows after some manipulation that the apparent viscosity is given by

$$\eta\{\kappa\} = \frac{f\{\kappa\}}{\kappa} = \frac{r}{2F\{r\}}, \tag{A 13}$$

where $\kappa = \gamma F\{r\}$. Equation (A 13) implies that the apparent viscosity must be constant.

Conversely, assume $f\{\kappa\} = \mu\kappa$. Then $f^{-1}\{\frac{1}{2}r\gamma\}/\gamma$ reduces to $r/(2\mu)$, which, for a given fluid, is clearly a function of r alone.

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